ESTIMATE OF THE CARRYING CAPACITY OF STRUCTURES UNDER ARBITRARY FLOW CONDITIONS

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This work is an elaboration of [1]. It is shown within the model of a St. Venant body that the lower limit of the maximum load can be found from a control problem that can be reduced to a variational problem. A method is proposed for obtaining simultaneously both the lower and upper limits. The condition under which this approach will give coinciding limits is found. The method is illustrated by examples of the calculation of composite shells.

Formulation of the Problem. We consider a structure made of a material whose behavior can be described by the model of an ideally plastic body. Let the yield condition have the form

$$f_1(\sigma) \cup f_2(\sigma) \cup \ldots \cup f_m(\sigma) = 1 \tag{1}$$

where σ is a vector consisting of the components of the stress tensor.

The loading is assumed to be uniparametric, i.e., the volume forces Q and the surface forces q are proportional to a single parameter t:

$$Q = Q_0 t,\ q = q_0 t\ (t > 0).$$

The equations of equilibrium can be written in the symbolic form

$$L\sigma = -Q_0 t\ (x \subset \omega),\ l\sigma = q_0 t\ (x \subset \Gamma), \tag{2}$$

where L and ℓ are linear operators, ω is the region occupied by the body, Γ is the surface of the body on which the loads are given, and x is the radius vector of a point.

The kinematic boundary conditions are assumed to be homogeneous and have the form

$$Ku = 0\ (x \subset \gamma). \tag{3}$$

Here K is a linear operator, u is the vector of displacements, and γ is the surface of the body on which the limits on the displacements are given. In what follows, the fields of the displacements are assumed to be kinematically possible, i.e., they are considered in spaces satisfying the condition (3).

Within the model of a St. Venant body there arises the problem of finding t_- and t_+ – the lower and upper limits, respectively, of the coefficient of the limiting load t_*, resulting in plastic failure of the structure. According to the static theorem [2-4], if for some Q and q the vector σ satisfies Eqs. (2) but does not exceed the limits of the yield surface, then $t = t_-$. If, however, the velocity field $\dot{u}$, satisfying the condition $K\dot{u} = 0$ on the surface γ has been found, then t_+ can be found [2-4].

Solution of the Problem. We consider first the situation when the left-hand side in the yield condition is a homogeneous function:

$$f(\sigma c) = c^n f(\sigma), \tag{4}$$

where f is convex. The case when the function f is quadratic was investigated in [1]. Here we study cases when f can be a nonquadratic function and when $n > 2$.

Writing $\sigma = t\sigma_0$ there follows from (2)

$$L\sigma_0 = -Q_0\ (x \subset \omega),\ l\sigma_0 = q_0\ (x \subset \Gamma). \tag{5}$$

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For $t = t_-$ the condition that the limits of the yield surface are not exceeded assumes the form

$$t_-^n f_{\max_x}(\sigma_0) = 1. \tag{6}$$

Hence one can see that in order to obtain the best lower estimate of t_* it is necessary to find a field σ_0 that minimizes $f\max_x(\sigma_0)$. The problem of finding σ_0 is replaced by the problem of finding the displacement field u and the field of elastic characteristics E, which are related to σ by a fictitious Hooke law:

$$\sigma_0 = E\varepsilon_0, \;\; \varepsilon_0(x) = Bu_0(x), \;\; E = \lambda(x)\; E_0(x). \tag{7}$$

Here ε_0 is a vector consisting of the components of the strain tensor; B is a linear differential operator; $\lambda(x)$ is a desired scalar; and E_0 is a symmetric matrix which will be found below from the condition $t_- = t_+$.

The computational operator $f\max_x(\sigma_0)$ is represented in the form

$$f_{\max_x}(\sigma_0) = \lim_{p\to\infty}\left[(\mathrm{mes}\,\omega)^{-1}\int_\omega f^p(\sigma_0)\,d\omega\right]^{1/p}, \tag{8}$$

and the problem of minimizing (8) is replaced by a variational problem of minimizing the function F under the restriction (5)

$$F_* = \min_{\lambda,u_0} F, \quad F = \int_\omega [f(\lambda E_0 B u_0)]^p\, d\omega. \tag{9}$$

It can be shown that this problem is equivalent to an isoparametric problem of minimizing F under the condition

$$\Pi = \int_\omega \lambda (Bu_0)^{\mathrm{T}} E_0 B u_0\, d\omega - \int_\omega Q_0^{\mathrm{T}} u_0\, d\omega - \int_\Gamma q_0^{\mathrm{T}} u_0\, d\Gamma = b, \tag{10}$$

where the index T designates transposition; b = const. Writing the condition for the Lagrangian $\Phi = F + \mu\Pi$ to be stationary in the form $\delta_\lambda \Phi = 0$, where δ indicates the variation operation, we obtain for λ

$$\lambda^{np-1} = (\mu/np)(Bu_0)^{\mathrm{T}} E_0 B u_0 / f^p(E_0 B u_0). \tag{11}$$

Writing $f_0 = f(\sigma_0)$ we obtain from the condition $\delta u_0 \Phi = 0$, taking into account Eq. (11),

$$\begin{aligned}&\int_\omega [\lambda (Bu_0)^{\mathrm{T}} E_0 B u_0 (\partial f_0/\partial\sigma_0)\, E_0 \delta B u_0 / n f_0]\, d\omega - \\ &- 2\int_\omega \lambda (Bu_0)^{\mathrm{T}} E_0 \delta B u_0 d\omega + \int_\omega Q_0^{\mathrm{T}} \delta u_0\, d\omega + \int_\Gamma q_0^{\mathrm{T}} \delta u_0\, d\Gamma = 0.\end{aligned} \tag{12}$$

The coefficients in front of δBu_0 in the first two integrals sum to $-\sigma_0^{\mathrm{T}}$:

$$\lambda (Bu_0)^{\mathrm{T}} E_0 B u_0 (\partial f_0/\partial\sigma_0)\, E_0/n f_0 - 2\lambda (Bu_0)^{\mathrm{T}} E_0 = -\sigma_0^{\mathrm{T}}. \tag{13}$$

The validity of Eq. (13) can be verified by multiplying Eq. (13) on the right by Bu_0, taking into account Eq. (7), and Euler's formula for homogeneous functions

$$(\partial f/\partial \sigma_0)^{\mathrm{T}} \sigma_0 = n f(\sigma_0),$$

Thus, Eq. (12) is a Lagrange variational equation equivalent to Eq. (5) and the problem Eqs. (9) and (5) is equivalent to Eqs. (9) and (10). The constant b [μ is expressed in terms of b through Eq. (10)] is arbitrary. Indeed, if Eq. (5) is represented by a single operator equation

$$D\lambda E_0 B u_0 = P_0$$

and its solution is written in the form

$$u_0 = (D\lambda E_0 B)^{-1} P_0, \tag{14}$$

then substituting Eq. (14) into Eqs. (5) and (7) shows that σ_0 does not depend on the amplitude λ.

In the limit $p \to \infty$ we represent the scalar $\lambda(x)$ in the form

$$\lambda = c/[f(E_0 B u_0)]^{1/n}. \tag{15}$$

Estimate from Above. Assume that the solution of the problem (9) and (10) has been found. In order to make an estimate from above, we take as the velocity field $v = \dot{u}$ and the strain-rate field $\xi = \dot{\varepsilon}$

$$v = u_0/\tau, \; \xi = B u_0/\tau \,,$$

where τ is a time constant.

Knowing ξ, we can, generally speaking, find the stress field σ_+ from the yield law:

$$\xi^{\mathrm{T}} = \nu\,[\partial f(\sigma_+)/\partial\sigma]^{\mathrm{T}}, \tag{16}$$

where ν is determined from the yield condition $f(\sigma_+) = 1$. The following approach can be used to calculate σ_+. Multiplying Eq. (16) on the right by σ_+ and using Euler's formula gives

$$\xi^{\mathrm{T}}\sigma_+ = \nu n f(\sigma_+) = \nu n.$$

According to the same formula of Euler

$$(n-1)\partial f(\sigma)/\partial\sigma = (\partial\,[\partial f/\partial\sigma]^{\mathrm{T}}/\partial\sigma)\sigma, \tag{17}$$

if f is twice continuously differentiable. Then Eq. (16) assumes the form

$$\xi = \nu\,[A(\sigma_+)/(n-1)]\sigma_+, \; A(\sigma) = \partial\,[\partial f/\partial\sigma]^{\mathrm{T}}/\partial\sigma. \tag{18}$$

In order to calculate the limits of the limiting load we propose using an iteration process. Thus, let the vector $\sigma_0 t_-$ be the argument in Eq. (18). Using the notation $A_0 = A(\sigma_0 t_-)$ and assuming A_0 is nonsingular, it follows from Eq. (18) that

$$\sigma_+ \cong A_0^{-1}\xi\,(n-1)/\nu. \tag{19}$$

The divisor ν is determined from the yield condition $f(\sigma_+) = 1$:

$$\nu = (n-1)\,[f(A_0^{-1}\xi)]^{1/n}.$$

Then

$$\sigma_+ \cong A_0^{-1}\xi/[f(A_0^{-1}\xi)]^{1/n}. \tag{20}$$

We can attempt to solve Eq. (18) by an iterative method as follows. Using as the argument of the matrix A the vector $\sigma_+^{(r-1)}$, where r is the number of the iteration, and $\sigma_+^{(0)} = t_-\sigma_0$, we obtain from Eq. (18)

$$\sigma_+^{r} = A_{(r-1)}^{-1}\xi/[f(A_{(r-1)}^{-1}\xi)]^{1/n}, \quad A_{(r-1)} = A(\sigma_+^{(r-1)}),$$

provided that $A_{(r-1)}$ are nonsingular matrixes.

If σ_+ has been found (exactly or approximately), then the upper limit of the limiting load can be calculated. According to the kinematic theorem, we have

$$t_+ = \int_\omega \sigma_+^{\mathrm{T}}\xi\,d\omega \Bigg/ \left(\int_\omega Q_0^{\mathrm{T}} v\,d\omega + \int_\Gamma q_0^{\mathrm{T}} v\,d\Gamma\right). \tag{21}$$

An iteration process for solving Eqs. (12) and (15) and calculating the limits t_- and t_+ can be constructed as follows. At the first step it assumed that $\lambda^{(1)} = 1$. The field $u_0^{(1)}$ is sought by solving the problem (3), (5), and (7) of the theory of elasticity. The lower limit t_- is found from the relation (6) and the value of t_+ is found with the help of Eq. (21). It should be noted that in the case when the formula (20) is employed, this is not strictly the upper limit, since the relation (20) is approximate. At the second step $\lambda^{(2)}$ is calculated using the formula (15) and the problem (3), (5), and (7) of the theory of elasticity is solved once again but with $\lambda = \lambda^{(2)}$, $t_-^{(2)}$, $t_+^{(2)}$ are sought once again, etc. As shown below for examples for a composite shell and in [1] for plates, the iteration process converges quite rapidly.

Condition for $t_- = t_+$. Assume that the solution of the problem (3), (5), and (7) of the theory of elasticity has been found, and assume that u_0 and λ are represented in the form

$$u_0 = \alpha\varphi(x),\ \alpha = \text{const},\ \lambda = c/[f(E_0B\varphi)]^{1/n}.$$

According to Eq. (4), we have

$$f(\sigma_0 t_-) = \lambda^n t_-^n \alpha^n f(E_0B\varphi) = t_-^n c^n \alpha^n. \tag{22}$$

Since $\max\limits_x f = 1$ necessarily, we have

$$t_- = |c\alpha|^{-1}. \tag{23}$$

In view of the fact that σ_0 satisfies the equations of equilibrium (5), the energy identity must also be satisfied:

$$\int\limits_\omega \sigma_0^{\mathrm{T}}\varepsilon_0\,d\omega = \int\limits_\omega Q_0^{\mathrm{T}}u_0\,d\omega + \int\limits_\Gamma q_0^{\mathrm{T}}u_0\,d\Gamma.$$

Substituting here σ_0 and ε_0, according to Eq. (7), and using Eq. (23) gives

$$\int\limits_\omega (B\varphi)^{\mathrm{T}} E_0B\varphi/[f(E_0B\varphi)]^{1/n}\,d\omega = \left(\int\limits_\omega Q_0^{\mathrm{T}}\varphi\,d\omega + \int\limits_\Gamma q_0^{\mathrm{T}}\varphi\,d\Gamma\right)t_-. \tag{24}$$

On the other hand, substituting into the formula for the estimate from above (20) and (21) the values $v = \alpha\varphi/\tau$, $\xi = \alpha B\varphi/\tau$, we obtain

$$\int\limits_\omega (B\varphi)^{\mathrm{T}} A_0^{-1}(B\varphi)/[f(A_0^{-1}B\varphi)]^{1/n}\,d\omega = \left(\int\limits_\omega Q_0^{\mathrm{T}}\varphi\,d\omega + \int\limits_\Gamma q_0^{\mathrm{T}}\varphi\,d\Gamma\right)\tilde{t}_+, \tag{25}$$

where $\tilde{t}_+$ is calculated with the help of the approximate formula (20).

Comparing Eqs. (24) and (25) shows that $\tilde{t}_+$ and t_- are identical if E_0 is found from the equations

$$E_0 = A_0^{-1}/\sigma_s,\quad A_0 = \partial\,[\partial f(t_-\lambda E_0B\varphi)/\partial\sigma]^{\mathrm{T}}/\partial\sigma \tag{26}$$

where σ_s is a constant with the dimension of stress.

It remains to show that $t_+ = \tilde{t}_+$ when Eqs. (12) and (15) are satisfied exactly. For this it is sufficient to verify that σ_-, found from Eqs. (12) and (15), satisfies the yield law (16). Indeed, as follows from Eqs. (22) and (23), the yield condition is satisfied, i.e., $f(\sigma_-) = 1$.

The yield law (16), substituting Eq. (17), assumes the form

$$\zeta = v\partial f(\sigma_-)/\partial\sigma = [v/(n-1)]\{\partial[\partial f(\sigma_-)/\partial\sigma]^{\mathrm{T}}/\partial\sigma\}\sigma_- = [v/(n-1)]A_0\sigma_-.$$

Substituting here $\sigma_- = \alpha t_-\lambda E_0B\varphi = \alpha t_-\lambda A_0^{-1}B\varphi/\sigma_s$ gives

$$\zeta = \{v\alpha t_-\lambda\tau/[(n-1)\sigma_s]\}\xi = \beta\xi, \tag{27}$$

where $\beta = [\cdot]$ is a scalar. Thus, as one can see from Eq. (27), the vectors ζ and ξ are collinear, i.e., ξ is orthogonal to the flow surface at the point σ_-, and hence σ_- and ξ satisfy the yield law (16). Therefore $\sigma_- = \sigma_+$ and $\tilde{t}_+ = t_+ = t_- = t_*$.

General Yield Condition. Assume that the yield condition has the form (1) and can be approximated by some inscribed piecewise-smooth surface, each piece of which is convex and is described by the equation

$$b_i(\sigma) = 1,\ i = 1,\ \ldots,\ p, \tag{28}$$

p is the number of pieces and b_i is a homogeneous function of degree 2n (for example, a quadratic function). Then the yield condition

$$b_1(\sigma) \cup b_2(\sigma) \cup \ldots \cup b_p(\sigma) = 1 \tag{29}$$

can be replaced by the single condition

$$g = \left(\sum_{i=1}^{p} [b_i(\sigma)]^k\right)^{1/k},\quad k \to \infty. \tag{30}$$

Indeed, if at least one of the conditions (29) is satisfied, then g = 1. Conversely, if g = 1, then at least one of the conditions (28) must be satisfied.

The function g, in contrast to the condition (29), describes a smooth surface. The validity of this substitution can be justified as follows. According to the static theorem, when making an estimate from below, it is only necessary that the conditions $b_i(\sigma_-) \leq 1$, i = 1, ..., p not break down in a statistically possible stress field σ_-. Since in the limit $k \to \infty$ the value $g \to \max\{b_1, \ldots, b_p\}$, then for $g(\sigma_-) \leq 1$ and $k \to \infty$ no value of b_i exceeds unity, i.e., the condition $b_i(\sigma_-) \leq 1$ remains valid.

The function $g(\sigma)$ is homogeneous. For this reason, the results obtained above are valid for it. Then

$$\lambda = c/[g(E_0Bu_0)]^{1/2n} = b_*^{1/2n},$$
$$b_* = \max\{b_1(E_0Bu_0), \ldots, b_p(E_0Bu_0)\}.$$

Here the fact that the other terms raised to the power k in the series in Eq. (30) makes an infinitesimal contribution.

From the geometric standpoint σ_+ is determined by seeking a vector representing a point on the surface (29), the normal to which at this point is collinear to the strain-rate vector ξ. If, however, ξ lies within the angle made by the normals to the surfaces $b_r = 1$ and $b_s = 1$ adjoining the edge, then σ_+ is set equal to the value of the vector σ_b representing a point on the edge (Fig. 1). The same procedure is used in the case of conical points [3, 4]. When the surface (29) is replaced by a smooth surface (30) the value of σ_g will be virtually identical to σ_b in the limit $k \to \infty$, as a result of which the error arising when the piecewise-smooth surface is replaced by a smooth surface will decrease as k increases.

In the case $\xi = 0$, i.e., when the stresses do not reach the flow surface, the vector σ_+ can be arbitrary, but, as one can see from Eq. (21), this does not affect the value of t_+.

Numerical Examples. As an illustration of the approach proposed above, Fig. 2 displays some computational results obtained by the finite-element method for thin composite shells of revolution, which are formed by symmetric winding of a ribbon at an angle $\pm\psi$ to the meridian along the geodesic lines of the median plane. The annular load $P = P_0t$ is directed along the axis of revolution and is applied to the edge with the smaller radius r_0. The results are presented for a conical shell with radius R_0 of the large freely supported edge with thickness $h_0 = 0.05r_0$ and with a winding angle ψ_0 at the supported edge. At other points of the meridian the thickness h and the angle ψ are related to the distance r to the axis of revolution by the relation [6]

$$r \sin\psi = R_0 \sin\psi_0,\ hr\cos\psi = h_0R_0\cos\psi_0.$$

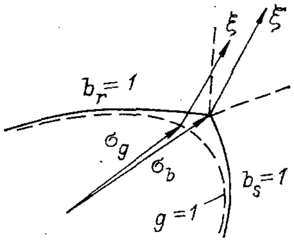

Fig. 1

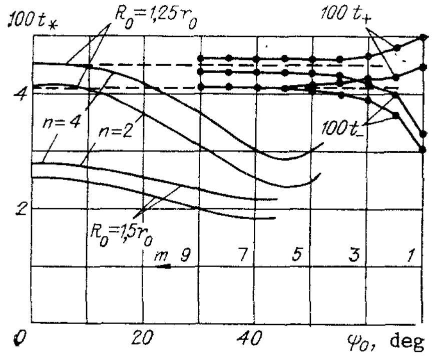

Fig. 2

The following yield conditions were taken for the ribbon:

$$(\sigma^{11}/\sigma_s^{11})^n + (\sigma^{22}/\sigma_s^{22})^n + (\sigma^{12}/\sigma_s^{12})^n = 1.$$

The results are presented for n = 2 and 4. It was assumed that $P_0 = \sigma_S R_0$, $\sigma_S{}^{11}/\sigma_S{}^{22} = 2$, $\sigma_S{}^{22}/\sigma_S{}^{12} = \sqrt{5}$, $t_* \cong t_{**} = (t_+ + t_-)/2$, and the height of the shell is r_0. The small increase in t_* for large values of ψ_0 is caused, first, by the fact that the thickness of the shell increases near the edge with the smaller radius r_0 as ψ_0 increases and, second, by the fact that $\sigma_S{}^{12} < \sigma_S{}^{22}$.

Figure 2 also displays plots of the convergence of the iteration process presented above. It is obvious that the value of t_{**} stabilizes very rapidly. The computational results also showed that the values of t_- and t_+ depend strongly on the degree of discretization of the region (the number of elements and points of integration over the thickness of the shell). For a high degree of discretization the numerical value of t_- can even be greater than t_+. This happens for two reasons. First, when approximate methods are employed for solving problems in the theory of elasticity the equations of equilibrium are, as a rule, not satisfied exactly. Second, when $f\max\limits_{x}(\sigma_0)$ is calculated, the peaks of the function f are "cut off" (smoothed) when numerical methods are employed.

In order for the computation to be stable and in order to achieve the best convergence, the ratio of the maximum value of λ to the minimum value must be limited. A similar condition must also be adhered to when calculating E_0 from Eq. (26). The results presented below were obtained for $\lambda_{max}/\lambda_{min} = 100$ and (with n = 4) $|\sigma^{ij}|_{max}/|\sigma^{ij}|_{min} = 100$ with E_0 calculated from Eq. (26). But, even with quite large deviations from the condition (26), the quantities t_+ and t_- are found to be quite close, and $t_{**} = (t_+ + t_-)/2$ stabilizes (with an error of about 1%) after 3-4 iterations.

Thus, the method proposed above makes it possible to obtain a bilateral estimate of the limiting load by solving a problem in the theory of elasticity whose elastic characteristics are determined in terms of the plasticity parameters.

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